MTH 516/616: Topology II

Problem Set

1. Show that every covering space of an orientable manifold is an orientable manifold.

2. Given a covering action of a group G on an orientable manifold M by orientation-preserving homeomorphisms, show that M/G is orientable.

3. We define the *connected sum* $M_1 \# M_2$ of two n-manifolds M_1 and M_2 to be the quotient space obtained by deleting the interior of closed n-balls $B_i \in M_i$ for $i = 1, 2$, and then identifying the $\partial B_i (\approx S^{n-1})$.

   (a) Show that if M_1 and M_2 are closed, then for $1 \leq i \leq n$,
   $$H_i(M_1 \# M_2) \cong H_i(M_1) \oplus H_i(M_2).$$

   (b) Show that if M_1 and M_2 are closed, then
   $$\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - \chi(S^n).$$

4. For a map $f : M \to N$ between two connected closed orientable n-manifolds, the *degree of* f is an integer d such that $f([M]) = d([N])$.

   (a) Show that for any connected closed n-manifold M, there is a map $f : M \to S^n$ of degree 1.

   (b) Show that a p-sheeted covering projection $p : M \to N$ has degree $\pm p$.

   (c) Show that for a degree 1 map $f : M \to N$, the induced homomorphisms at the level of fundamental group and homology are both surjective.

5. Show that direct limits of exact sequences in exact. More generally, if $\{C_\alpha, f_{\alpha\beta}\}$ is a directed system of chain complexes, then show that
   $$H_n(\varinjlim C_\alpha) = \varinjlim H_n(C_\alpha).$$

6. Show that for all $\alpha \in C_k(X; R)$, $\varphi \in C^\ell(X; R)$, and $\psi \in C^m(X; R)$
   $$(\alpha \frown \varphi) \frown \psi = \alpha \frown (\psi \smile \chi).$$
   Consequently, deduce that the cap product makes $H_*(X; R)$ a right $H^*(X; R)$-module.

7. Show that $H^0_c(X; G) = 0$, if X is path-connected and noncompact.

8. If M is a connected compact orientable n-manifold, a homeomorphism $f : M \to M$ is *orientation preserving* if f_* takes the fundamental class to itself, and *orientation reversing* otherwise. Use the cup product to show that every homeomorphism $f : \mathbb{C}P^2 \to \mathbb{C}P^2$ orientation preserving.

9. Show that if a connected closed orientable manifold M of dimension $2k$ has $H_{k-1}(M; \mathbb{Z})$ torsion-free, then $H_k(M; \mathbb{Z})$ is also torsion-free.

10. If $p : (\widetilde{X}, \widetilde{A}, \widetilde{x}_0) \to (X, A, x_0)$ is a covering space with $\widetilde{A} = p^{-1}(A)$, then show that $p_* : \pi_n(\widetilde{X}, \widetilde{A}, \widetilde{x}_0) \to \pi_n(X, A, x_0)$ is an isomorphism for $n > 1$.

11. Show that if $\varphi : X \to Y$ is a homotopy equivalence, then the induced homomorphisms $\varphi_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ are isomorphisms for $n > 1$.